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Properties of weighted structured scale-free networks

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Abstract. A simple model for weighted structured scale-free (WSSF) networks is proposed. The growth dynamics of the network is based on a naive weight-driven deactivation mechanism which couples the establishment of new active vertices and the weights' dynamical evolution. Simulations show that all the interesting statistical properties of the generated network (vertices degree, vertices strength and links weight) display good right-skewed distribution observed in many realistic systems. Particularly, if the constant bias factor in deactivation probability is appropriately chosen, a power law distribution $P(k) \sim k^{-\gamma}$ for vertices total degree k with the exponent $\gamma = 3$ is obtained. As a survey of the model, the epidemic spreading process in WSSF networks is studied based on the standard *susceptible-infected* (SI) model. The spreading velocity reaches a peak very quickly after the infection outbreaks which is similar to the case of infection propagation in other heterogeneous networks; and in the long time propagation it decays approximately with an exponential form.

PACS. 89.75.Hc Networks and genealogical trees – 87.23.Kg Dynamics of evolution – 05.70.Ln Nonequilibrium and irreversible thermodynamics

1 Introduction

Complex networks have attracted an increasing interest in the last few years [1]. The main reason is that they play an important role in the understanding of complex behaviors in real world networks, including the structure of language [2,3], scientific collaboration networks [4,5], the Internet [6,7] and World Wide Web [8,9], power grids [10], food webs [11,12], chemical reaction networks [13], metabolic [14] and protein networks [15], etc. The highly heterogeneous topology of these networks is mainly reflected in two characters, the small average path lengths among any two vertices (small-world property) [16] and a power law distribution (scale-free property), $P(k) \sim k^{-\gamma}$ with $2 \leq \gamma \leq 3$, for the probability that any vertex has k connections to other vertices [17]. Furthermore, the aging and weight of vertices and links are of particular interest [10,18,19]. In citation networks, papers cease to receive links because their contents are outdated or summarized in review papers, which are then cited instead; and also some famous papers would likely be cited more than those ordinary ones. The developed metaphors considering the effect of vertices aging and links weight to the growth of the network are the so-called structured scale-free networks [20,21] and weighted evolving networks [10,19,22]. The introduction of vertices aging mechanism and links weight evolving mechanism provide us with a profound view on understanding and characterizing realistic systems.

In the study of complex networks, a good example is to inspect the effect of their complex features on the dynamics of epidemic spreading [23]. It is easy to foresee that the characterization and understanding of epidemic dynamics on these networks can find immediate applications to a large number of problems, such as computer virus infections, transmission of public opinion, etc. However, so far, most studies of epidemic spreading just focus on unweighted networks, and a detailed inspection of epidemic spreading in weighted networks is very rare. In this paper, we first proposed a simple model for weighted structured scale-free (WSSF) network based on weight-driven deactivation mechanism which couples the establishment of new active vertices and the weights' dynamical evolution. It was found that all the interesting statistical properties of the generated network (*vertices degree, vertices strength and links weight*) display good right-skewed distribution observed in many realistic systems. As a particular survey of the model, the standard *susceptible-infected* (SI) model [24] for epidemic spreading was studied in WSSF networks. The propagation velocity displays driving growth tendency and reaches a peak very quickly after the infection outbreaks, which is similar to the case of infection propagation in other heterogeneous networks [25].

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In the long time propagation, the velocity decays approximately with a exponential form different from the studied power-law decay in weighted scale-free networks [26].

2 Previous weight-driven evolving models

Weighted networks can be described by a matrix w_{ij} specifying the weight on the edge connecting the vertices i and j, with $i, j = 1, ..., N$ where N is the size of the network ($w_{ij} = 0$ if the vertices i and j are not connected). Then the strength of the vertex i can be defined as [19,27]

$$s_i = \sum_{j \in \mathcal{V}(i)} w_{ij}, \tag{1}$$

where the sum runs over the set $\mathcal{V}(i)$ of neighbors of i. Recently, Barrat, Barthélemy and Vespignani (BBV) [28] have proposed a model for the evolving of weighted network when new edges and vertices are continuously added into the network while causing dynamic behavior of the weights. Their model starts from an initial number of completely connected vertices N_0 with a same assigned weight w_0 to each links. At each subsequent time step, addition of a new vertex n with m_0 edges and corresponding modification in weights are implemented by the following two rules: (i) The new vertex n is attached at random to a previously existing vertex i according to the probability distribution

$$\Pi_{n \to i} = \frac{s_i}{\sum_j s_j}. \tag{2}$$

This rule relaxes the usual degree preferential attachment, focusing on a strength driven attachment in which new vertices connect more likely to vertices handling larger weights. (ii) A total induced increase δ in strength s_i of the ith vertex is distributed among it's nearest neighbors $j \in \mathcal{V}(i)$ according to the rule

$$w_{ij} \to w_{ij} + \delta \frac{w_{ij}}{s_i}. \tag{3}$$

More recently, Pandya [29] argued that this second rule, though could be one possibility, does not follow the same mechanism of the first rule. For the case of the worldwide airport network, the first rule can be described as "busy airports get busier" according to the dynamics driven factor s_i. The second rule, however, can be instead described by "busy routes get busier" since the route i to j having more traffic as indicated by w_{ij} would handle larger portion of the induced traffic δ given by $\delta \frac{w_{ij}}{s_i}$. That does not necessarily mean that the airport j, in the neighbor of i, with largest value for w_{ij} is also the airport with maximum strength or traffic in comparison with other neighboring airports of i. Pandya rewrite the equation (3) as

$$w_{ij} \to w_{ij} + \delta \frac{s_j}{\sum_{k \in \mathcal{V}(i)} s_k} \tag{4}$$

where $\mathcal{V}(i)$ indicates set of all neighboring airports of i and $k \neq n$. The last term of equation (4) indicates that it is more probable that the induced traffic would go towards the airport j handling maximum traffic s_j among the neighboring airports $\mathcal{V}(i)$, which is then in consistent with the mechanism of the first rule of BBV. Moreover, One can easily see in BBV model that $\lim_{t\to\infty} s_i(t) \to \infty$ in the limit of long time, which is not in accordant with most realistic condition. The deactivation mechanism of vertices of the growing network introduced by Klemm [20] can well avoid the case of infinite increasing of the vertices strength.

3 Weight-driven deactivation model

Motivated by some beautiful previous work [20,28,29] and the statements indicated in the preceding sections, we construct our model to study the self-organization of WSSF networks. The model describes the growth dynamics of a network with directed links. Rather than the degree-dependent deactivation dynamics of the vertices developed in reference [20] generating structured scale-free networks (SSF), our model is based on the weight-driven deactivation dynamics of the vertices, which can be constructed as the following steps.

First, start from an initial seed of m_0 vertices completely connected by undirected links with assigned weight $w_0 = 1.0$. By k_i' we denote the in-degree of vertex i, i.e., the number of links pointing to vertex i, and by s_i' the total induced strength by in-degree links of vertex i. Each vertex of the network can be in two different states: active or inactive. As the initial condition we let all the m_0 vertices active. At each time step, a new vertex n is added with m_0 links that are attached to the previously existing m_0 active vertices. Each new added link is assigned weight w_0 and induce a total strength increasing $w_0 + \delta$ to the linked active vertex. The additional weight δ will be distributed among the out-degree links of the aim vertex according to the rule

$$w_{ij} \to w_{ij} + \delta \frac{s_j'}{\sum_{k \in \mathcal{V}(i)} s_k'}. \tag{5}$$

For the case of the worldwide airport network, equation (5) indicates that it is more probable that larger capacity airports would handle more additional traffic which comes from the neighbor airport i. Again, the induced total strength of i's neighbors can also be further redistributed among the weights of the neighbors of neighbors of airport i, and so on the neighbors of neighbors of neighbors. For simplicity, we consider only the first order rearrangement of the strength, which means that each time step a new vertex added into the network will increase only the strength of it's first and second nearest neighbors. The new added vertex is always in the *active* state first. It receives links from subsequently generated vertices until it is deactivated. Remembering that at each time step only m_0 vertices in the network are permitted active and all the others are inactive, we would deactivate one of the $m_0 + 1$ active vertices after the new active vertex added to the network. To perform this, we assume that the probability rate P of deactivation decreases with the total induced strength of the vertex. Then the deactivation probability of a vertex i with strength s'_i can be written as

$$P(s'_i) \propto \frac{\gamma - 1}{a + s'_i} \tag{6}$$

where $a > 0$ is a constant bias factor and the normalization factor is defined as $\gamma - 1 = (\sum_{l \in \mathcal{A}} \frac{1}{a+s'_l})^{-1}$. The summation runs over the set $\mathcal{A}$ of the currently $m_0 + 1$ active vertices.

Note that the larger strength a vertex possesses, the more difficult for it to be deactivated, or in other words, the more easier for it gaining new links. For the case of the citation network, equation (6) means that the famous paper cited mostly would less probability to be "forgotten" [20]. Also for the case of the worldwide airport network, equation (6) indicates that it is less probable that a new airport would not build a new airline with those airports handling more traffic. And for the case of the internet, equation (6) implies that requests are more likely sent to those servers handling more task or possessing better capability to ask for senior servers, which can be reflected by the summation of the additional distributed weight among the vertices' out-links.

4 Structural properties

Following reference [20], the distribution $N(k')$ of the in-degree k' can be obtained analytically for the model defined above, considering the continuous limit of k'. Let us first derive the distribution $p^{(t)}(k')$ of the in-degree of the active vertices at time t. For $k' > 0$, the time evolution is determined by the following master equation

$$p^{(t+1)}(k'+1) = (1 - P(k'))p^{(t)}(k'), \tag{7}$$

where a and γ have been defined in Section 3 of the model definition and $P(k')$ is the deactivation probability of a vertex with in-degree k'. The boundary value $p(0)$ is a constant reflecting the constant rate of new vertices with initial $k' = 0$. Noting that the deactivation model would generate networks with chain like structure (implying that the newest added vertices would barely have an influence on the oldest vertices' strength) and each added new link will increase the quantity of induced strength of $1 + \delta$ to the linked vertex, in the large time limit, we can get an approximate relation between s' and k' of the vertices, i.e., $s' \approx (1 + \delta)k'$, then we obtain $P(k') \simeq P\left(\frac{s'}{1+\delta}\right) = (1 + \delta)P(s')$, where $P(s')$ is the deactivation probability of a vertex with strength s'. Substituting them into equation (7), we yield

$$p^{(t+1)}(k'+1) = \left(1 - \frac{\gamma - 1}{\frac{a}{1+\delta} + k'}\right) p^{(t)}(k'). \tag{8}$$

Assuming that the fluctuations of the normalization $\gamma - 1$ are small enough, such that γ may be treated as a constant, the stationary case $p^{(t+1)}(k') = p^{(t)}(k')$ of equation (8) yields

$$p(k'+1) - p(k') = -\frac{\gamma - 1}{\frac{a}{1+\delta} + k'} p(k'). \tag{9}$$

Treating k' as continuous we write

$$\frac{\mathrm{d}p}{\mathrm{d}k'} = -\frac{\gamma - 1}{\frac{a}{1+\delta} + k'} p(k'), \tag{10}$$

and obtain the solution

$$p(k') = b\left(\frac{a}{1+\delta} + k'\right)^{-\gamma+1}, \tag{11}$$

with appropriate normalization constant b. In case the total number n of vertices in the network is large compared with the number m_0 of active vertices, the overall in-degree distribution $N(k')$ can be approximated by considering the inactive vertices only. Thus $N(k')$ can be calculated as the rate of change of the degree distribution $p(k')$ of the active vertices. We find

$$N(k') = -\frac{\mathrm{d}p}{\mathrm{d}k'} = c\left(\frac{a}{1+\delta} + k'\right)^{-\gamma} \tag{12}$$

with $c = (\gamma - 1)(\frac{a}{1+\delta})^{\gamma-1}$. The exponent γ is obtained from a self-consistency condition obtained from the average connectivity

$$m_0 = c\int_0^\infty \frac{k'}{(\frac{a}{1+\delta} + k')^\gamma} dk', \tag{13}$$

which gives

$$\gamma = 2 + \frac{a}{m_0(1+\delta)}. \tag{14}$$

Thus the exponent γ depends only on the ratio $a/m_0(1+\delta)$. In Figure 1a, MC data of the l.h.s. of equation (12) as a function of $k' + \frac{a}{1+\delta}$ is plotted under different values of $a = 2m_0(1+\delta)$ and $0.5m_0(1+\delta)$, and power law decay behavior is obtained with exponent $\gamma = 4.02 \pm 0.05$ and 2.48 ± 0.05 respectively, which is expected as equation (14). For convenience, we rewrite the l.h.s. of equation (12) as $N(k' + \frac{a}{1+\delta})$. Then, if we choose the value of the constant bias $a = m_0(1+\delta)$, equation (12) is no other than the probability distribution of vertices total degree $k = (m_0 + k')$ of the network. In Figure 1b, we plot the total degree distribution of the network with $m_0 = 10$ and different values of $\delta = 0.0, 0.5, 1.0, 2.0$. The total size of the network is $N = 10^6$. A power law distribution $P(k) \sim (k)^{-\gamma}$ with best fitted exponent $\gamma = 2.98 \pm 0.05$ is obtained, which again is well in agreement with the analytic result equation (14). For other values of the constant bias a, the distribution of the overall degree of the networks are right-skewed.

Notice that each new link added to the network will induce $w_0 + \delta$ strength increase to the aim vertex, which indicates that the vertices with larger in-degree would

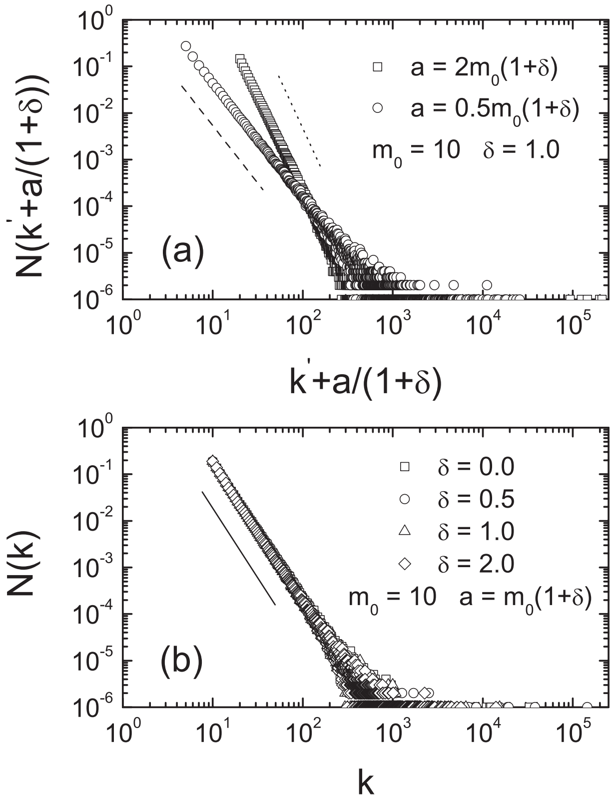

Fig. 1. (a) the log-log representation of the l.h.s. of equation (12) as a function of $k' + \frac{a}{1+\delta}$ under different values of a, the dot line and dash line are guides to the eye with power law decay exponent 4.0 and 2.5 respectively. (b) illustration of the probability distribution of vertices total degree of a weighted structured scale-free network with $w_0 = 1$, $m_0 = 10$, and $a = m_0(1+\delta)$, the straight line is a power law k^{-3}. All the experiment networks have a total number of vertices $N = 10^6$.

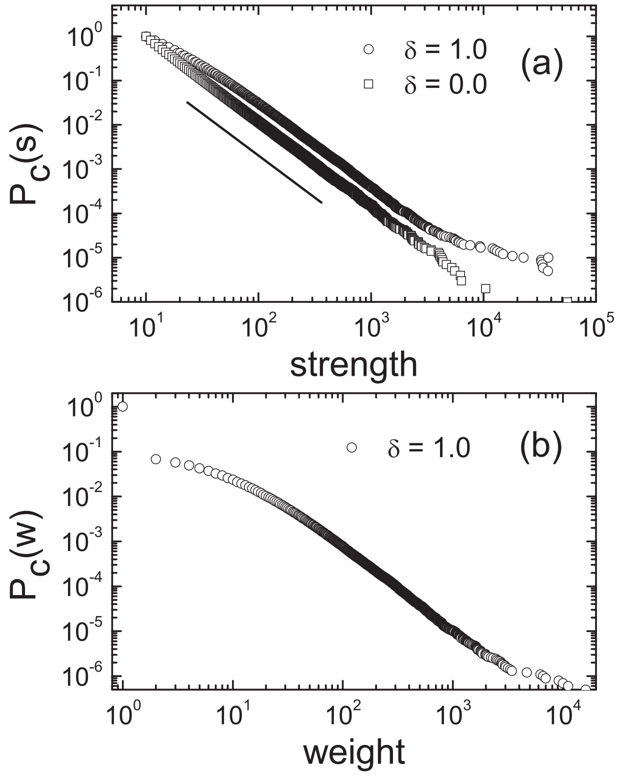

Fig. 2. Illustration of the probability distribution of cumulative vertices strength (a), and cumulative links weight (b) of a weighted structured scale-free network with a total number of vertices $N = 10^6$, $m_0 = 10$, $w_0 = \delta = 1.0$, and $a = m_0\,(1+\delta)$. Both properties display good right-skewed distribution. By comparison, the case of $\delta = 0.0$ is also plotted in (a), which recovers the usual degree-dependent deactivation model, and the strength decays with a power-law form (solid line).

likewise have larger induced strength. According to our evolving network rule, the vertices with larger induced strength have more probability to gain new links, then the usual degree preferential attachment is reasonably recovered. This means that the right-skewed character of the network, such as the vertices total strength, also of the links weight, will retain. In order to decrease the statistical fluctuation, we report the cumulative probability distribution of the these two properties in Figure 2. The results are well expected showing good right-skewed character, which is reasonably in agreement with the condition of many realistic systems [4,5,10–12].

5 Epidemic spreading

To study the dynamics of infectious diseases spreading in weighted networks, we shall study the standard *susceptible-infected* (SI) model [24]. In this model individuals can only exist in two discrete states: susceptible (or healthy) and infected. The model can be described in terms of the densities of susceptible and infected individuals, $s(t)$ and $i(t)$, respectively, then $s(t)+i(t) = 1$. Each individual is represented by a vertex of the network and the links are the connections between individuals along which the infection may spread. In weighted networks, according to reference [26], the spreading rate can be defined as

$$\lambda_{ij} = \left(\frac{w_{ij}}{w_{max}}\right)^{\theta} \tag{15}$$

at which susceptible individual i acquire the infection from the infected neighbor j, where θ is a positive constant and w_{max} is the largest value of w_{ij} in the network. In this model, infected individuals remain always infective, an approximation that is useful to describe early epidemic stages in which no control measures are deployed.

We start simulations by selecting one vertex randomly and assuming it is infected. The disease will spread in the network in according with the rule of equation (15). In Figure 3 we plot the density of infected individuals versus Monte Carlo (MC) time in WSSF networks with $N = 10^4, \delta = w_0 = 1.0, m_0 = 10$ and $\theta = 0.5, 0.7, 0.8$. Note that $\frac{w_{ij}}{w_{max}} \leq 1$, the smaller value of θ means more quickly the infection spreads.

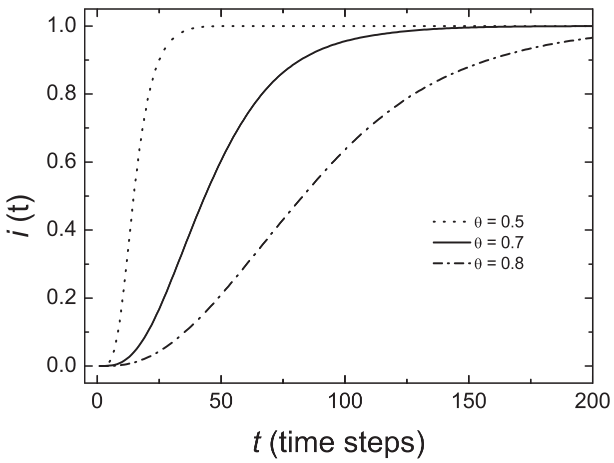

Fig. 3. Density of infected individuals versus MC time in WSSF networks with $N = 10^4$, $m_0 = 10$, $\delta = w_0 = 1.0$ and $a = m_0(1 + \delta)$. The date are averaged over 500 realizations.

For the SI model of epidemic spreading, it is expected that all the individuals will be infected in the limit of long time $\lim_{t\to\infty} i(t) = 1$. We will study the spreading velocity at the outbreak moment which is defined as

$$V_{inf}(t) = \frac{di(t)}{dt} \approx \frac{i(t) - i(t-1)}{N}. \quad (16)$$

We account the number of newly infected vertices at each time step and report the spreading velocity in Figure 4. The spreading velocity goes up to a peak in a short time that similar to the spreading process on weighted scale-free network [26] and unweighted heterogeneous network case [25], which leave us very short response time to develop control measures. By comparison, in the inset of Figure 4a, the case of $\delta = 0.0$ (which recovers the original SSF network proposed in Ref. [20]) is also plotted, and the velocity fluctuates around some plateau values just as for a one-dimensional regular lattice, which reflects the one-dimensional chain structure of the SSF networks. In fact, according to the rule of spreading, the case of $\delta = 0.0$ correspond to the limit case of $\theta \to 0$ for $\delta \neq 0.0$. Similarly, the character of multi-peak in Figure 4a also reflects the chain like structure of the generated WSSF networks. To decrease the fluctuation, in Figure 4b, the experiment data averaging 500 realizations of a network of 10^4 populations with approximate values of θ are reported. One can see in the long time propagation that the power-law decay of the velocity, which has been studied in reference [26] in the weighted scale-free network does not arise in our WSSF networks, in which the velocity decays more likely with an exponential form after the "peak time".

6 Conclusions

In this paper, we have studied a simple evolving model for weighted structured scale-free networks. The growth dynamics of the network is governed by a naive weight-driven deactivation mechanism. The deactivation probability is proportional to the inverse of the vertices strength induced by their in-degree links, which characterize the vertices' capability of obtaining further links. All interesting properties of the generated network display good right-skewed distribution character, which have been found very common in most realistic systems. And with the simple setting of the model the degree distribution displays a power law where the exponent can be obtained analytically. Furthermore, the spreading process of infectious diseases in WSSF networks has been investigated by the simplest SI model, showing that the propagation velocity reaches a peak value very quickly at the initial infection period and then decays approximately with an exponential form.

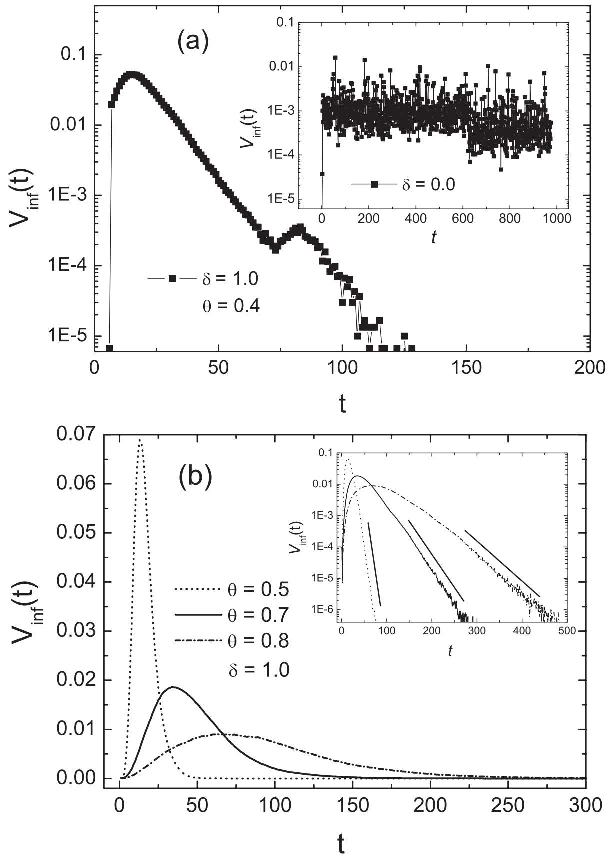

Fig. 4. Spreading velocity versus MC time steps in WSSF network with $m_0 = 10$, $\delta = w_0 = 1.0$ and $a = m_0(1 + \delta)$. The total size of the network is $N = 3 \times 10^5$ (a) and $N = 10^4$ (b). By comparison, in the inset of (a), the case of $\delta = 0.0$ is also plotted and the velocity fluctuates around some plateau values reflecting the chain like structure of the SSF networks. In the inset of (b), we report the date on linear-log representation. It is approximate that the velocity decays exponentially in the large time limit. The date in (b) are averaged over 500 realizations.

The model we have explored, however, is possibly the simplest one in the class of weight-driven growing networks. One can notice that in our model the out-degree of the vertices remain unchanged in the whole evolving period, which is likely unreasonable for realistic conditions. There exists a series of improvements can be made further. For instance, rather than remaining a constant, δ

is a variable for different vertices depended on their degree or strength; the deactivation mechanism would differ from equation (6) with a more complex form to mimic in a detailed fashion particular networked system, etc. All of these are deserve to make further studies.

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